A HELLY-NUMBER FOR k-ALMOST-NEIGHBORLY SETS

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ABSTRACT

It is possible to prove the following "Helly-type" theorem for k-almost-neighborly sets.

THEOREM. *$A \subseteq R^d$ is k-almost-neighborly if and only if every $2d + 1$ member subset of A is k-almost-neighborly. Moreover, if $A \subseteq$* bdry conv *A, then A is k-almost-neighborly if and only if every 2d member subset of A is k-almost-neighborly. Each bound is best possible.*

Let A be an arbitrary subset of R^d with dim conv $A = d$. Then A is k-almost-neighborly if and only if for every k member subset B of A, conv $B \subseteq$ bdry conv A. Since dim conv $A = d$, there is a set of $d + 1$ points of A whose convex hull is d dimensional. Therefore, the definition of k-almost-neighborly is meaningful only for $1 \leqq k \leqq d$. If A is k-almost-neighborly, then A is j-almost-neighborly for $1 \leqq j \leqq k$. Clearly if $A \cap \text{int conv } A \neq \phi$, then A is not k-almost-neighborly for any $1 \leqq k$.

It is easily proved (and in fact appears as an exercise in Grünbaum [2]) that a subset A of R^d is k-almost-neighborly if and only if every $2d + k$ member subset of A is k-almost-neighborly. Grünbaum has conjectured that the bound might be lowered to $2d + 1$. The following theorem shows that $2d + 1$ is indeed the correct Helly-number.

THEOREM. *Let A be an arbitrary subset of R^d. Then A is k-almost-neighborly if and only if every $2d + 1$ member subset of A is k-almost-neighborly. Moreover, if $A \subseteq$* bdry conv *A, then A is k-almost-neighborly if and only if every 2d member subset of A is k-almost-neighborly. Each bound is best possible.*

The proof will require the following result, which is a special case of theorem D of Bonnice-Reay [1].

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LEMMA. *Let* $A \subseteq R^d$, $p \in \text{int conv } A$. *Let* $B \subseteq A$ *be the vertex set of a maximal simplex for which* $p \in \text{rel int conv } B, j = \dim \text{conv } B$. *Then there is a subset* C *of* A *containing* B *with* $d + 1 \leqq |C| \leqq 2d - j + 1 \leqq 2d$ and $p \in \text{int conv } C$.

PROOF. Certainly if A is k-almost-neighborly, then every subset of A is k-almost-neighborly.

To prove the converse, consider two cases.

Case 1. If $A \nsubseteq \text{bdry conv } A$, then select some point p in $A \cap \text{int conv } A$. Using the Lemma, select C in A such that $d + 1 \leqq |C| \leqq 2d$ and $p \in \text{int conv } C$. Then $|C \cup \{p\}| \leqq 2d + 1$ yet $p \notin \text{bdry conv } (C \cup \{p\})$. Thus $C \cup \{p\}$ is not 1-almost-neighborly and cannot be k-almost-neighborly. Therefore A is not k-almost-neighborly. We have proved that if $A \nsubseteq \text{bdry conv } A$, neither A nor every $2d + 1$ member subset of A is k-almost-neighborly and the theorem is satisfied.

Case 2. Henceforth assume $A \subseteq \text{bdry conv } A$. We use a contrapositive argument. Assume that A is not k-almost-neighborly and let B denote a k member subset of A for which conv $B \nsubseteq \text{bdry conv } A$. Thus conv B contains some point p interior to conv A. Reduce to a subset B_0 of B such that $p \in \text{rel int conv } B_0$ and B_0 is the vertex set of a maximal simplex. Using the Lemma, if $j = \dim \text{conv } B_0$, select a d dimensional subset C of A containing B_0 with $d + 1 \leqq |C| \leqq 2d - j + 1 \leqq 2d$ and $p \in \text{int conv } C$. Certainly $|B_0| = j + 1 \leqq k$. Since $p \in \text{conv } B_0$, conv B_0 $\nsubseteq \text{bdry conv } C$, C is not $(j + 1)$-almost-neighborly and certainly not k-almost-neighborly, completing the proof.

The following examples show the bounds in case 1 and 2 are best possible.

EXAMPLE 1. Let e_i be the point in R^d whose ith component is 1 and whose other components are zero. Let A denote the points e_i, $1 \leqq i \leqq d$, together with their negatives and the origin p. Then $A \nsubseteq \text{bdry conv } A$, each $2d$ member subset of A is 1-almost-neighborly while A is not.

EXAMPLE 2. For A in example 1, let $B = A \sim \{p\}$. Then each $2d - 1$ member subset of B is 2-almost-neighborly while B is not.

REFERENCES

1. W. Bonnice and J. R. Reay, *Relative Interiors of Convex Hulls*, Amer. Math. Soc. Proc., **20**, (1969), 246–250.

2. B. Grünbaum, *Convex Polytopes*, New York, 1967.

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